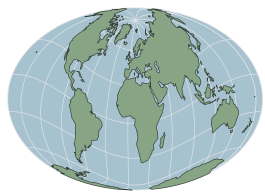
– 7 –

Euler's Map Theorem

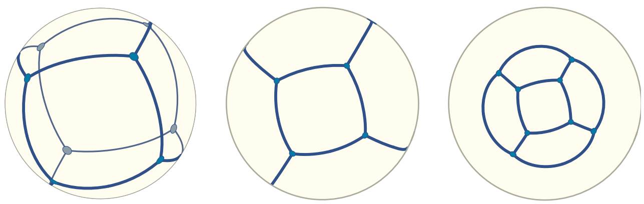
We've made some powerful deductions from Euler's Theorem that $V - E + F = 2$ for maps on the sphere. Now we'll prove it!



This is really quite familiar—we've all seen maps of the Earth in the plane.

Proof of Euler's Theorem

We can copy any map on the sphere into the plane by making one of the faces very big, so that it covers most of the sphere.



We'll think of this big face as the ocean, the vertices as towns (the largest being Rome), the edges as dykes or roads, and ourselves as barbarian sea-raiders! (See Figure 7.1.)

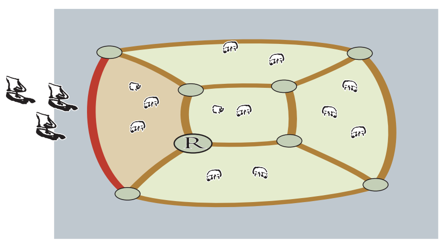
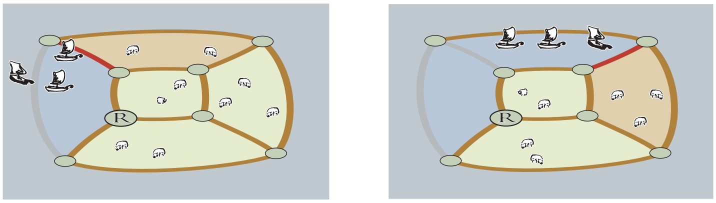


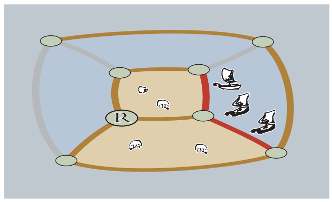
Figure 7.1. Our prey.

(opposite page) Like all maps on the sphere, this beautiful map (signature ***532**) has $V + F - E = 2$.

In this new-found role, our first aim is to flood all the faces as efficiently as possible. To do this, we repeatedly break dykes that separate currently dry faces from the water and flood those faces. This removes just $F - 1$ edges, one for each face other than the ocean, by breaking $F - 1$ dykes.



Deleting an edge decreases the number of edges by 1 and also decreases the number of faces by 1, so $V - E + F$ is unchanged.



We next repeatedly seek out towns other than Rome that are connected to the rest by just one road, sack those towns, and destroy those roads. (See Figure 7.2.)

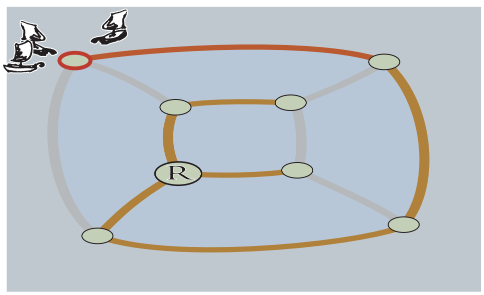
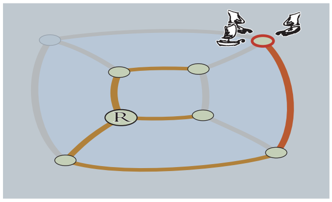
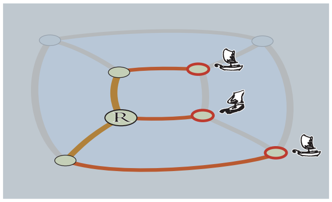
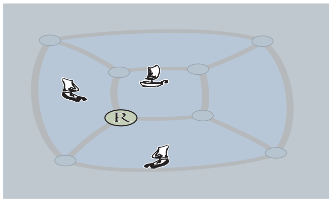


Figure 7.2. Our raid continues!







Deleting a vertex and the edge joining it to the rest of the tree does not change $V - E + F$ either, so $V - E + F$ must always have been 2.

We have sacked $V - 1$ towns by destroying $V - 1$ roads, one for each town other than Rome. The number of edges in the original map must therefore have been $(F - 1) + (V - 1) = V + F - 2 = E$. Therefore, $V + F - E = 2$, proving Euler's Theorem.

(Did we sack every town other than Rome? Yes; an unsacked town furthest from Rome would have two paths back to Rome, which however must enclose some dry fields, a contradiction. Did we destroy all remaining roads? Yes; an undestroyed road must be between unsacked towns, which must both be Rome; but then again it must enclose some dry fields.)

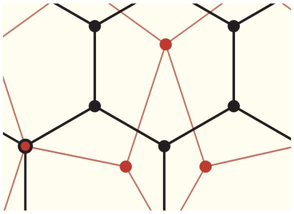
We have tacitly assumed that each face is a topological disk, and we will continue to suppose this. We have also taken for granted some intuitively obvious facts about the topology of the sphere whose formal proofs are surprisingly difficult.

The number 2 is Euler's characteristic number for the sphere. Every surface has such a number.

The Euler Characteristic of a Surface

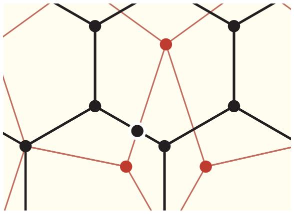
Theorem 7.1 *Any two maps on the same surface have the same value of $V - E + F$, which is called the* Euler characteristic *for that surface.*

We prove that any two maps on the same surface have the same Euler characteristic $V - E + F$ by considering a larger map obtained by drawing them both together. We shall suppose that no two edges meet more than finitely often, pushing the maps around a bit if necessary.

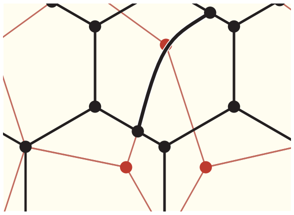


We first draw one map in black ink, the other in red pencil. Then we gradually ink in parts of the pencil map, noticing that $V-E+F$ does not change. The following figures show the first few steps of this process for a pair of maps.

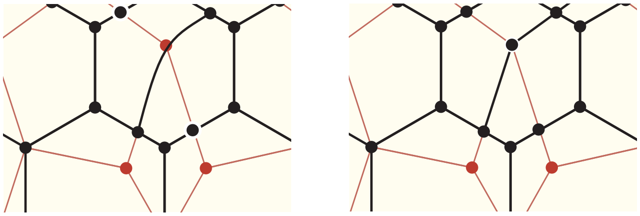
Inserting a vertex. V increases by 1, E increases by $2-1=1$, so $V-E+F$ increases by $1-1+0=0$.

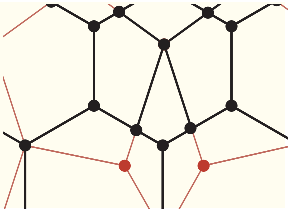


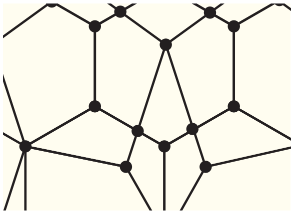
Inserting an edge. E increases by 1, F increases by $2-1=1$, so $V-E+F$ increases by $0-1+1=0$.



We can continue to make these insertions, gradually inking in the entire figure and not changing $V-E+F$:





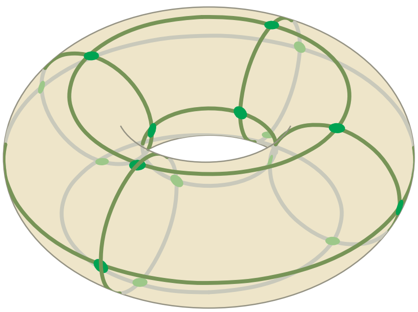


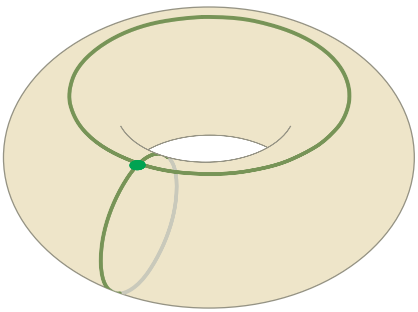
This argument shows that the characteristic number $V - E + F$ for the compound map is the same as that for the originally black map. Equally, it's the same for the originally red map! Therefore, those two original maps must have had the same characteristic.

The Euler Characteristics of Familiar Surfaces

Let us work out a few examples.

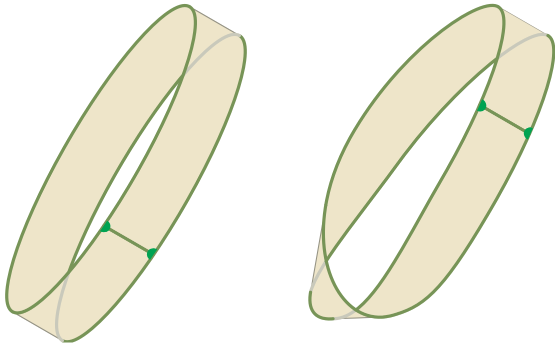
The Euler characteristic of a torus is 0.





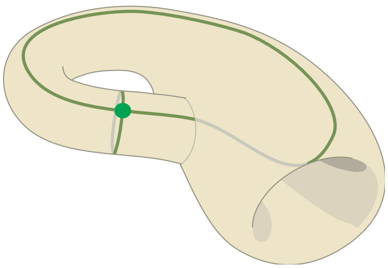
The map on the left has 16 vertices, 32 edges, and 16 faces, so $V-E+F = 16-32+16 = 0$. The map on the right is much simpler: it has just 1 vertex, 2 edges, and 1 face, so $V - E + F = 1 - 2 + 1 = 0$. The theorem tells us that we can use either map to work out the characteristic.

The Euler characteristic of an annulus or Möbius band is 0.



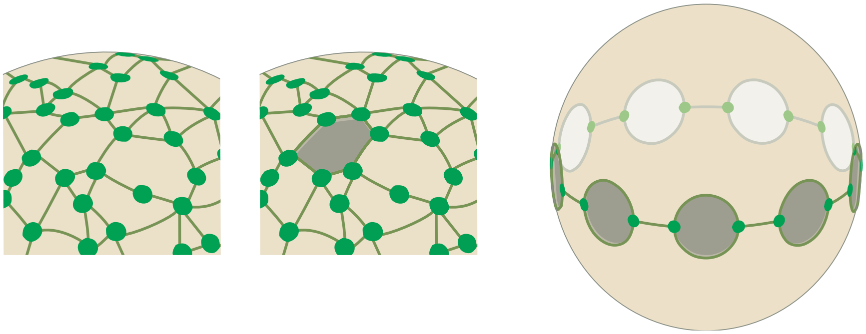
On the left, we see a map on an annulus, on the right a map on a Möbius band. Both maps have 2 vertices, 3 edges, and 1 face, and so $V - E + F = 0$.

The Klein bottle also has Euler characteristic 0.



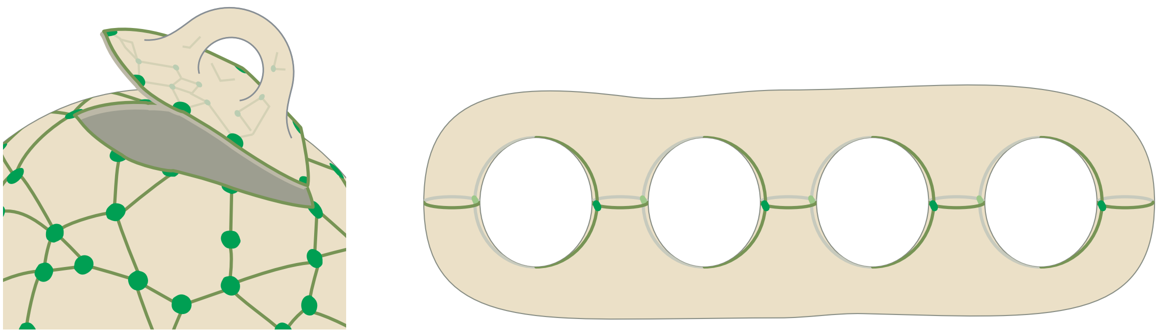
The Klein bottle, a one-sided, boundary-less surface, also has Euler characteristic 0. Again, we choose a map with just 1 vertex, 2 edges, and 1 face, yielding $V - E + F = 1 - 2 + 1 = 0$.

A sphere with n holes punched in it has Euler characteristic $2 - n$.



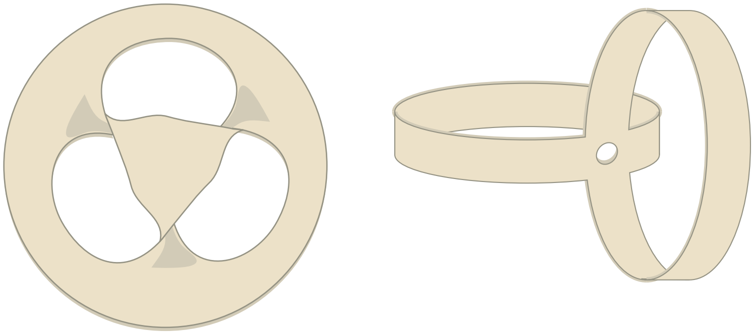
We may see this easily by taking a map on the sphere that has a great many more than n faces. If we delete n non-adjacent faces, we have kept V and E the same but decreased F by n. Consquently, the Euler characteristic will be n less than that of a sphere: $2 - n$. (In fact, punching n holes in any surface will always decrease the Euler characteristic by n.) Alternatively, we may systematically design a map specifically for this surface. On the right above, we see a map with $2n$ vertices, $3n$ edges, and 2 faces, and so $V - E + F = 2n - 3n + 2 = 2 - n$. As we will see in Chapter 8, a disk is topologically equivalent to a sphere with one hole in it, and so a disk has Euler characteristic 1.

An n-fold torus has Euler characteristic $2 - 2n$.



An n-fold torus is a surface obtained from a sphere by adding n handles, or equivalently n tunnels. We make it by deleting n faces from a sphere and then attaching n handles. Each handle is just a torus with a (very large) hole punched in it and will contribute $0 - 1$ to the total Euler characteristic. Each hole punched in the sphere will contribute -1. So the net result is that the Euler characteristic of an n-holed torus is $2 - 2n$. Or we may design a map specifically for this surface, with $2n$ vertices, $4n$ edges, and 2 faces: $V - E + F = 2n - 4n + 2 = 2 - 2n$.

Two mystery surfaces with Euler characteristic –2.



Here we have two mystery surfaces with $V - E + F = -2$. Both have two boundaries and are two-sided; in Chapter 8, we will learn that they then must be the same surface, topologically. In the meantime, you might try to decide for yourself whether this is obvious!

Where Are We?

In this chapter we have shown that for the sphere the Euler characteristic is 2 and more generally that the value of $V - E + F$ depends only on the surface on which a map is drawn and not on the map itself. This supports the proof of the Magic Theorem in Chapter 6, which in turn supports the enumeration of symmetry types in Chapters 2–5.

In the next chapter we shall classify all possible surfaces, which will show us all the forms an orbifold could possibly take and will help us conclude that we've enumerated the signatures of all possible symmetry types.