Arcball Interface

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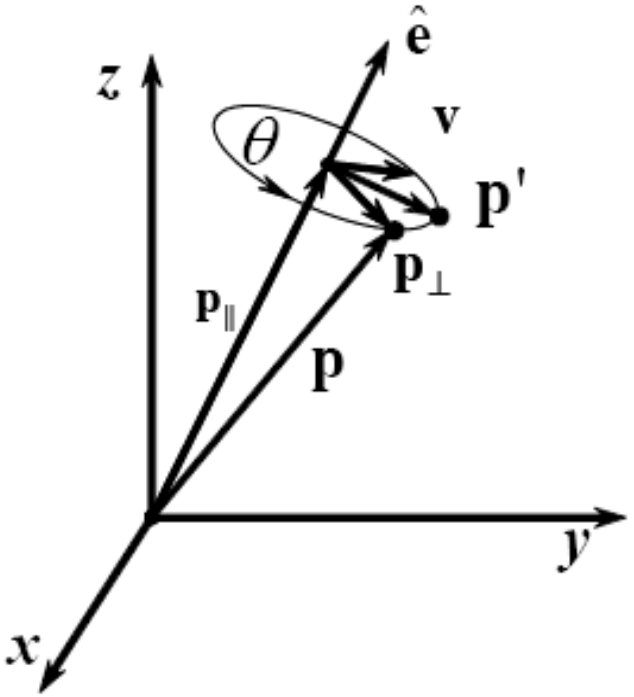
Definition

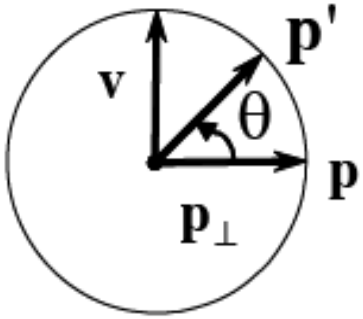
- **Goal:** implementation of an algorithm for rotating an object, by using the Arcball paradigm
- **Input:** mouse movement
- **Output:** a matrix to rotate, in a natural way, a 3D object

Writing Rotation About $\hat{e}$

p' described in orthogonal base $\hat{e}, p_\perp, v, \left\|\hat{e}\right\| = 1$



$$\mathbf{v} = \hat{\mathbf{e}} \times \mathbf{p}_\perp$$



$$p = p_\perp + p_\parallel$$

$$p' = R(p_\perp) + R(p_\parallel)$$

$$v = \hat{e} \times p_\perp$$

$$p' = p_\parallel + (\cos\theta)\, p_\perp + (\sin\theta)\, v$$

$$p_\parallel = (\hat{e} \cdot p)\hat{e}$$

$$p_\perp = p - (\hat{e} \cdot p)\hat{e}$$

$$v = \hat{e} \times p_\perp = \hat{e} \times (p - (\hat{e} \cdot p)\hat{e}) = \hat{e} \times p - (\hat{e} \cdot p)\hat{e} \times \hat{e} = \hat{e} \times p$$

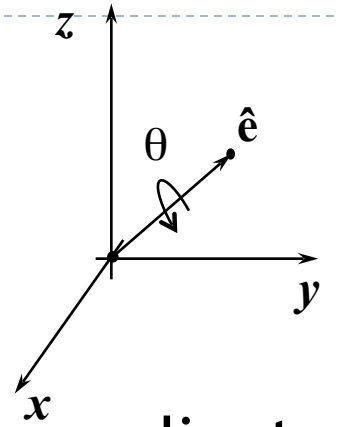
$$p' = (\hat{e} \cdot p)\hat{e} + (\cos\theta)(p - (\hat{e} \cdot p)\hat{e}) + (\sin\theta)(\hat{e} \times p)$$

$$p' = (\cos\theta)\, p + (1 - \cos\theta)(\hat{e} \cdot p)\hat{e} + (\sin\theta)(\hat{e} \times p)$$

Rotation Matrix

- Rotation about an axis.

ê – axis of rotation
θ – rotation angle

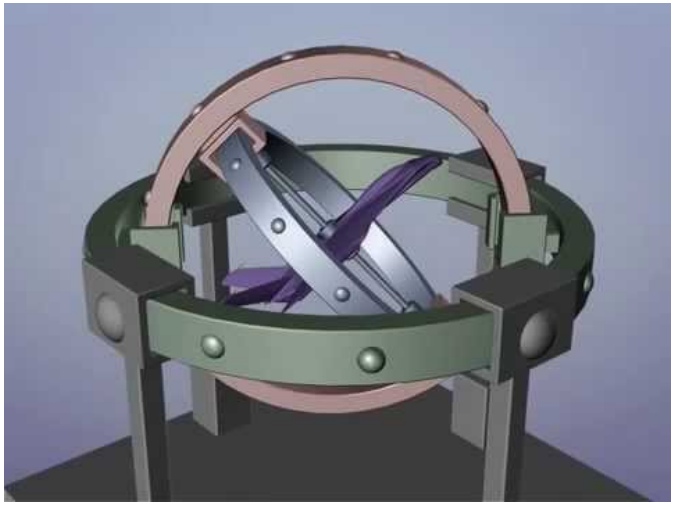


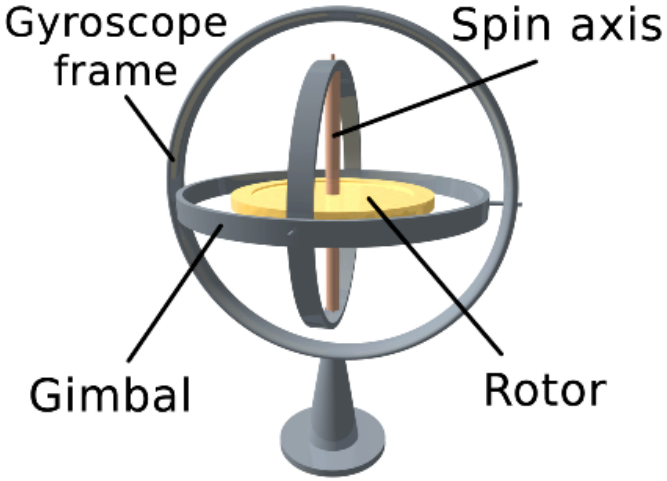
  - OpenGL - glRotatef(θ, ex, ey, ez)
- Rotation matrix, after changing coordinate system from $\hat{e}, p_{\perp}, v$ to x,y,z:
  - $p'(p) = (\cos\theta)p + (1-\cos\theta)(\hat{e}\cdot p)\hat{e} + (\sin\theta)(\hat{e}\times p)$
  - Columns of M are given by: p'(1,0,0), p'(0,1,0), p'(0,0,1)

$$\mathbf{M} = \begin{bmatrix} \cos\theta + (1-\cos\theta)e_x^2 & e_y e_x(1-\cos\theta) - e_z\sin\theta & e_z e_x(1-\cos\theta) + e_y\sin\theta & 0 \\ e_x e_y(1-\cos\theta) + e_z\sin\theta & \cos\theta + (1-\cos\theta)e_y^2 & e_z e_y(1-\cos\theta) - e_x\sin\theta & 0 \\ e_x e_z(1-\cos\theta) - e_y\sin\theta & e_y e_z(1-\cos\theta) - e_x\sin\theta & \cos\theta + (1-\cos\theta)e_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Euler Angles in Animation

- Gimbal Lock





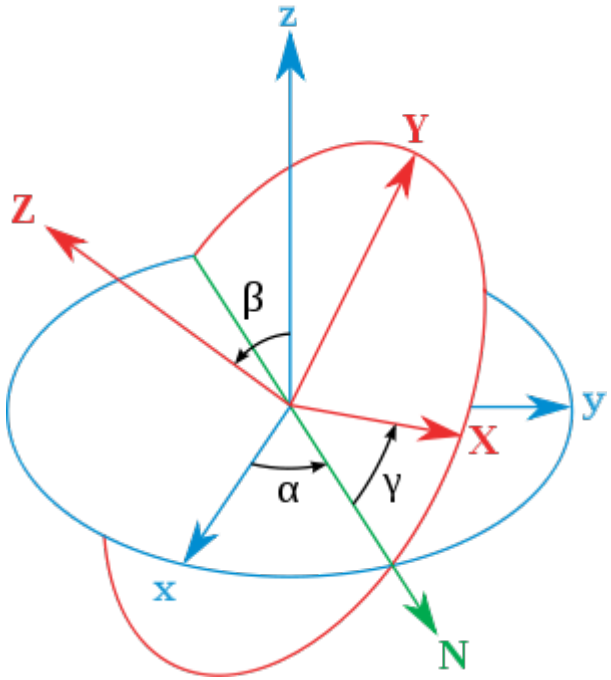
- Is the loss of one degree of freedom in a 3D, 3-gimbal mechanism, which occurs when the axes of two gimbals are driven into a parallel configuration
  - "locking" the system into rotation in a degenerate two-dimensional space.
  - https://www.youtube.com/watch?v=zc8b2Jo7mno

Gimbal lock on Apollo 13

- “13, Houston. We see you getting close to gimbal lock there. We'd like you to bring up all quad Cs on MAIN A, quad C-1, C-2, C-3, C-4 on MAIN A, and also bring B-3 and B-4 up on MAIN A.”
- Apollo used an **I**nertial **M**echanism **U**nit for navigation, same as airliners do today.
  - Three gyroscopes. One points up and down, one right and left, the other fore and back. Accelerometers attached to these record accelerations in each direction.
  - Given a starting point, such a system can record where you are in relation to that point at any time. Each of these gyroscopes will have a "null point“, where the gyro lines up perfectly with its gimbals. If all three enter this state at once, the platform instantly loses it knowledge of where it is. It goes into "gimbal lock".
  - http://www.universetoday.com/119984/13-more-things-that-saved-apollo-13-part-9-avoiding-gimbal-lock/

Euler Angles Between Two Reference Frames xyz to XYZ

- α represents a rotation around z.
- N (line of nodes) is the orientation of X after the first elemental rotation (x').
  - B represents a rotation around x'.
- The third rotation occurs about Z. Hence, Z = z".
  - γ represents a rotation around z".
- z-x'-z" (intrinsic rotations)
  - α and γ in [−π, π] and
  - β in [−π/2, π/2].
  - B = 0 and plane XY coincides with plane xy -> gimbal lock.



Gimbal Lock in CG

- In fact, this is just an analogy (a good one?), and the problem only occurs if Euler angles are used.
  - Some systems (Blender, Maya) use just three numbers, R_x, R_y, R_z, instead of a 4 dimensional array (quaternion) to store an orientation.
  - A general rotation R is then calculated, by multiplying these three matrices (or, some say, applying three "rolls"):

$$R(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 & 0 \\ 0 & -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  - The animator uses the parameterization above to set up an arbitrary orientation.
    - An x-roll, then a y-roll and finally a z-roll
    - https://sundaram.wordpress.com/2013/03/08/mathematical-reason-behind-gimbal-lock-in-euler-angles/

Euler Angles and Gimbal Lock

- If the y-roll happens to be 90º, then the "*lock*" occurs:

$$R(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} c_2c_3 & c_2s_3 & -s_2 & 0 \\ s_1s_2c_3 - c_1s_3 & s_1s_2s_3 + c_1c_3 & s_1c_2 & 0 \\ c_1s_2c_3 + s_1s_3 & c_1s_2s_3 - s_1c_3 & c_1c_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, s_i = \sin\theta_i, c_i = \cos\theta_i$$

$$R(\theta_1,\frac{\pi}{2},\theta_3) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ \sin(\theta_1-\theta_3) & \cos(\theta_1-\theta_3) & 0 & 0 \\ \cos(\theta_1-\theta_3) & -\sin(\theta_1-\theta_3) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, s_2 = 1, c_2 = 0$$

- The animator has stumbled upon a **singularity** in the parameterization space (θ_2 is gone).
  - θ_1 *and* θ_3 are associated with the same degree of freedom.
  - http://en.wikipedia.org/wiki/Euler_angles

How to Avoid

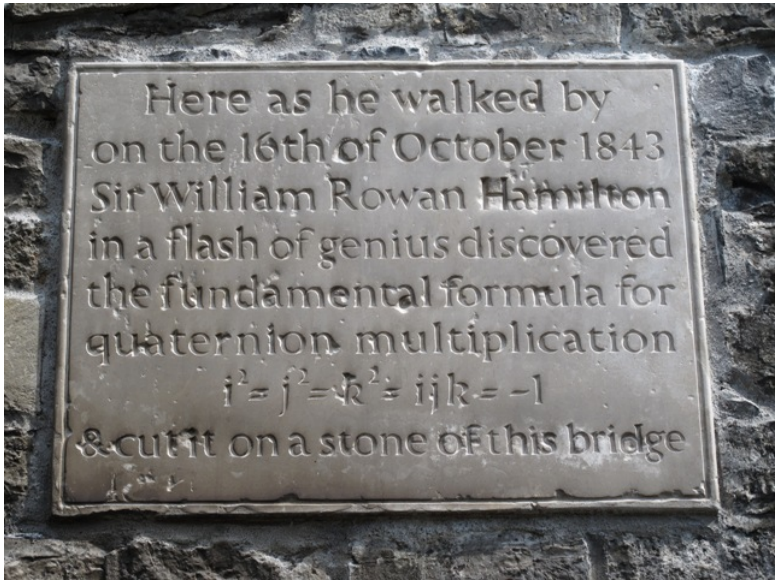
- Lock occurs because Euler angles are being changed, in a fixed order, one at a time, so axes can line up.
- If all three angles are changed simultaneously, the lock does not occur
  - But interpolation from one key orientation to another is not unique
- Another possibility is using a “fourth gimbal”, not allowed to align, which take us to a…
- Better paradigm: quaternions.

Sir William Rowan Hamilton

- Hamilton, in 1830, tried to generalize complex numbers to R^3 (a complex volume, with two imaginary axes)
  - $a + ib$, $i^2 = -1$
- He realized in 1843 that it was impossible on R^3 (triples of numbers are not closed under multiplication), but not on R^4 or R^8
  - $q = a + bi + cj + dk = (a,v)$, $i^2 = j^2 = k^2 = ijk = -1$
  - $ij = k$, $jk = i$, $ki = j$, $kj = -i = ji = -k$, $ik = -j$
- The multiplication of two quaternions is a quaternion
  - Closed under multiplication => mathematical group.
  - Quaternion multiplication is not commutative! $q_1.q_2 \neq q_2.q_1$

Broome Bridge in Dublin

- Greatest Irish mathematician ever (?)



- https://www.youtube.com/watch?v=mHVwd8gYLnI&nohtml5=False

Quaternions

- Complex Numbers on R^4
  - q = a + b**i** + c**j** + d**k**
  - $q = (s, v)$,
    - s – real component and
    - v – vector representing the imaginary component
- Simplifies the calculation of rotations about an axis

$$q = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{e}} \right)$$

ê – unit vector (rotation axis)
θ – rotation angle

- Rotations using product of quaternions
  - Rotation of a point p about an axis v
    - $R(p) = q\, p\, q^{-1}$
      - $p = (0, r)$ - point represented as a pure quaternion
      - $q = (s, v)$ - rotation (angle and axis) represented as a quaternion

Properties of Quaternions

- Multiplication and Addition

$$q_1 = (s_1, v_1),\ q_2 = (s_2, v_2)$$

$$q_1 q_2 = (s_1 s_2 - v_1 \cdot v_2,\ s_1 v_2 + s_2 v_1 + v_1 \times v_2)$$

$$q_1 + q_2 = (s_1 + s_2,\ v_1 + v_2)$$

- Conjugate and reciprocal: $q^{-1} = \dfrac{\bar{q}}{\|q\|^2}$

$$\bar{q} = (s, -v)$$

$$q\bar{q} = (s, v)(s, -v) = s^2 + \|v\|^2 = s^2 + x^2 + y^2 + z^2 = \|q\|^2$$

- Norm: $\|q\| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{s^2 + \|v\|^2}$
- Rotation: take a pure quaternion p = (0, r) and a **unit** quaternion q=(s,v), where $q\bar{q} = 1$, and define:
  - $R(p) = q\ p\ q^{-1}$

Orientations as Quaternions

$$q = (\cos(\theta), \sin(\theta)\hat{\mathbf{e}}),\ q^{-1} = (\cos(\theta), -\sin(\theta)\hat{\mathbf{e}}),\ \|\hat{\mathbf{e}}\| = 1,\ p = (0, r)$$

$$R_q(p) = qpq^{-1} = (0, (\cos^2\theta - \sin^2\theta)r + 2\sin^2\theta(\hat{e}\cdot r)\hat{e} + 2\cos\theta\sin\theta(\hat{e}\times r)) =$$
$$= (0, (\cos 2\theta)r + (1-\cos 2\theta)(\hat{e}\cdot r)\hat{e} + \sin 2\theta(\hat{e}\times r))$$

- This is exactly the same equation in transparency 4
  - aside from a factor of 2, and *p* written as *r*
- Therefore, orientations can be parameterized as:

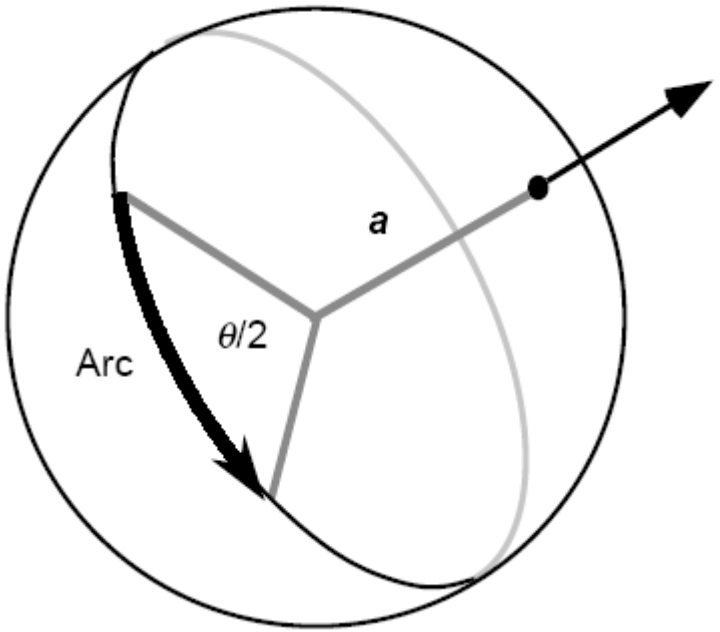
$$q = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{e}}\right),\ \|\hat{\mathbf{e}}\| = 1$$

Moving In and Out Quaternion Space

- Unit quaternions are closed under multiplication
  - Multiplication of two unit quaternions is a unit quaternion
  - Pile up rotations really nice, with no gimbal locks!
- Go from a general rotation matrix to quaternion and vice versa.
- Taking a unit quaternion $q = \left(\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{e}} \right), \|\hat{\mathbf{e}}\| = 1$ and performing $q()q^{-1}$ is the same as applying a rotation matrix:
  - double quat[4]
  - double mat[4][4]
  - mat2quat (double* mat, double* q)
  - quat2mat (double* q, double* mat)
- Code can be found in reference [2]

ArcBall

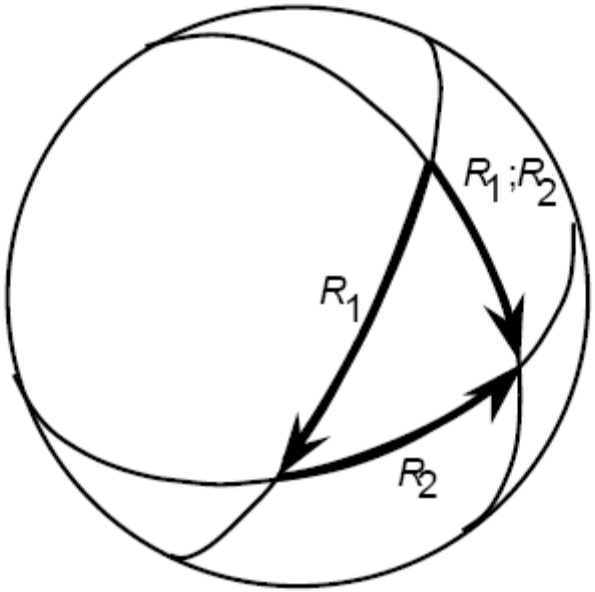
- **Definition:** interface for rotating a 3*D* object, by using just a mouse



Arc Interpretation

a – rotation axis

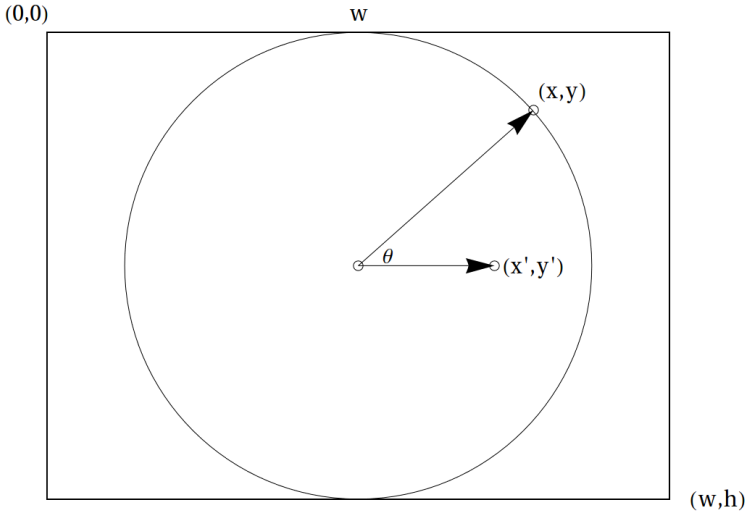
θ – rotation angle



Arc Combination

Sphere As a Virtual Trackball

1. Consider screen coordinates from (0,0) to (w,h).
2. Mouse coordinates (x,y) are used to sample a function, z(x,y):



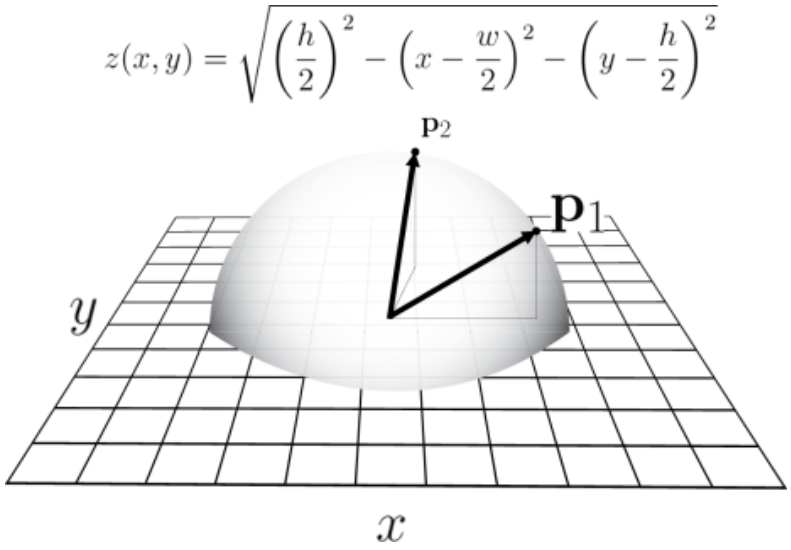
3. Points inside the circle are mapped onto a sphere, while points outside are mapped onto plane z=0.

Hemisphere Inscribed in Screen Area

- Circle centered at the origin:

$$x^2 + y^2 + z^2 = R^2 \Rightarrow z = \sqrt{R^2 - x^2 - y^2}$$

- Circle centered at (w/2, h/2):

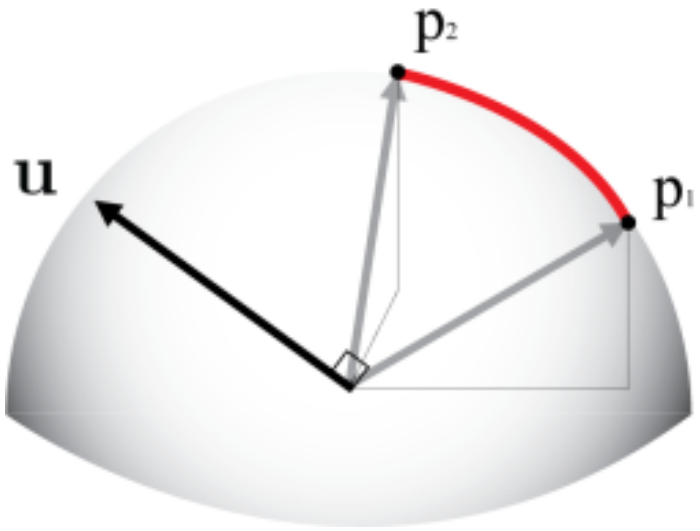


Code [1]

```
radius = MIN(w/2,h/2)
x = (x1 - w/2)/radius;
y = (h/2 - y1)/radius;
r = x*x + y*y;
if (r > 1.0) {
    s = 1.0/sqrt(r);
        x *= s;
        y *= s;
        z = 0.0;
}       else
        z = sqrt(1.0 – r);
```

Angle-axis Transformation

- Mouse dragged from $p_1 = (x, y, z(x, y))$ to $p_2 = (x', y', z'(x', y'))$
- Rotation angle: $\theta = \arccos\left(\frac{p_1 \cdot p_2}{||p_2|| \cdot ||p_1||}\right)$
- Rotation axis: $u = \frac{p_1 \times p_2}{||p_2|| \cdot ||p_1||}$



Rotations as Two Reflections

- Practical implementations use two reflections instead of a rotation
  - Product of two reflections in intersecting planes is equivalent to a rotation.
  - The axis of this rotation is the line of intersection of the planes, while the angle of rotation is twice the angle between the two planes.

Example

- Put two mirrors at 90° to each other and put an object in front of them.
- The reflection on the second mirror will appear to have been rotated by 180° relative to the original object.
- The rotation angle 180° is twice the angle between mirrors, which is 90° in this example.
- https://sureshemre.wordpress.com/2014/07/19/a-rotation-equals-two-reflections/

simple-rotator.js

- setView = function(viewDirVec, viewUpVec, viewDistance)
  - Create view matrix using (u_x, u_y, u_z) as column vectors
    - viewDirectionVector --> u_z
    - viewUpVector x u_z --> u_x
    - u_z x u_x --> u_y
- getViewMatrix = function()
  - Return a rotated view matrix (after dragging the mouse)
- function toRay(x,y)
  - Create a ray from the center of the screen to the point on the sphere, which projects onto (x,y)
  - Sphere origin is the center of screen, and radius is the minimum (width, height)

simple-rotator.js

- function doMouseDrag(evt)
  - Create two rays: ray_1,ray_2
  - map them to world coordinates, by using the view matrix inverse (transpose)
  - applyTransvection(ray_1,ray_2);
- function applyTransvection(e_1, e_2)
  - Rotate view matrix (u_x, u_y, u_z) around (e_1 x e_2), by applying two reflections on each vector.

- http://math.hws.edu/eck/cs424/notes2013/webgl/skybox-and-reflection/simple-rotator.js

Conclusions

- Euler angles make objects move unexpectedly (skew strangely) between key frames
  - Gimbal lock can occur in any order of rotation
  - Best order for camera in Maya is:
    - y (yaw) -> x (pitch) -> z (roll)
    - (only locks looking straight up or down)
- Quaternions are the appropriate paradigm for rotation interpolation, and they avoid gimbal lock
- One can add, multiply and divide two quaternions, but multiplication does not commute ($p.q \neq q.p$).
- Multiplication of unit quaternions correspond to matrix multiplication (rotation composition)

Conclusions

- Arcball does not require quaternions to be implemented
  - Very intuitive and only requires a mouse
- Replaces all those sliders, for changing angles, for a click and pull.
- Qt timer class provides a tick event that can be used to re-apply a quaternion, on and on, to produce an animation.
  - Or requestAnimationFrame(render) in html5/javascript.

I Am Done for Today

Thank you all!!

References

- [1] *ARCBALL: A User Interface for Specifying Three-Dimensional Orientation Using a Mouse* - Ken Shoemake
- [2] *Advanced Animation and Rendering Techniques* - Alan Watt
- [3] https://braintrekking.wordpress.com/2012/08/21/tutorial-of-arcball-without-quaternions/